AP STATISTICS
TOPIC 9: EXPONENTIAL FUNCTIONS

PAUL L. BAILEY

Abstract. Let a be a positive real number. Most students who have attended a Precalculus class probably understand the meaning of a^x when x is a positive integer, and possibly if x is a negative integer. They tend to not be able to say the meaning of a^x when x is rational, and none will be able to define a^x when x is irrational.

To continue in our study of Calculus and/or Statistics, we need to discuss exponential functions and their properties, and to define the famous yet elusive constant e. Sadly, it is common practice to ignore the mathematics behind the study of these ideas, and simply tell students "just do this". It is an unacceptable act of malpractice to do so in a mathematics class, where critical thinking is the most important obtainable skill.

Thus, it is necessary to determine the meaning of a^x, where x is irrational, to actually define e, and to prove the important properties that will be used. The cleanest way to do this requires usage of integration, but their are good reasons to introduce exponentials and logarithms earlier in a Calculus course. This note is intended to motivate the definition of a^x and e, and to develop the properties or these types of functions which will be used as we move forward.

1. Sequences

Let X be a set. A *sequence* in X is a function $x : \mathbb{N} \to X$; that is, a sequence is a function whose domain is the set of positive integers. We write x_n instead of $x(n)$ to mean the n^{th} term in the sequence. We write (x_n) to denote the entire sequence. Think of $(x_n) = (x_1, x_2, x_3, \dots)$ as a tuple of infinite length.

We are interested is sequences or numbers; that is, sequences whose codomain is $\mathbb{R}$; we may think of such a sequence as an ordered list of numbers or infinite length. Our development requires the limits of sequences, which we define now.

Let $X = \mathbb{R}$ and let (x_n) be a sequence in $\mathbb{R}$; that is, (x_n) is a sequence of real numbers. Let $L \in \mathbb{R}$. We say that (x_n) *converges* to L, and that the *limit* of (x_n) is L, if

$$\text{for every } \epsilon > 0 \text{ there exists } N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow |x_n - L| < \epsilon.$$

In this case, we write $\lim\limits_{n\to\infty} x_n = L$.

That is, (x_n) converges to L as n goes to infinity if x_n gets closer to L as n gets bigger, and if you make n large enough, you can get x_n as close to L as you want it to get.

For example, let $x_n = \dfrac{1}{n}$. Since x_n gets smaller as n gets larger, and we can make x_n as close to zero as we want by making n large enough, we see that $\lim\limits_{n\to\infty} x_n = 0$.

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2. Exponents

Let a be a positive real number, and let x be a real number. We ask, what is the meaning of a^x?

2.1. **When x is a positive integer.** Let $n = x$, and assume that n is a positive integer. Then a^n is defined to mean the product of n numbers whose value is a:

$$a^n = \underbrace{a \times \cdots \times a}_{n \text{ times}}.$$

From this, we obtain two significant properties.

(E1) $a^{m+n} = a^m \cdot a^n$
(E2) $(a^m)^n = a^{mn}$

To see this, write

$$a^{m+n} = \underbrace{a \times \cdots \times a}_{m+n \text{ times}} = \underbrace{a \times \cdots \times a}_{m \text{ times}} \times \underbrace{a \times \cdots \times a}_{n \text{ times}} = a^m \times a^n.$$

and

$$(a^m)^n = (\underbrace{a \times \cdots \times a}_{m \text{ times}})^n = \underbrace{(\underbrace{a \times \cdots \times a}_{m \text{ times}}) \times \cdots \times (\underbrace{a \times \cdots \times a}_{m \text{ times}})}_{n \text{ times}} = \underbrace{a \times \cdots \times a}_{mn \text{ times}} = a^{mn}.$$

We wish to extend the meaning of a^x so that it is defined for any real number x, in such a way that the properties **(E1)** and **(E2)** remain true.

2.2. **When $x = 0$.** Consider the case when $x = 0$. We multiply a times a^0; whatever a^0 means, if property **(E1)** is to remain true, we have

$$aa^0 = a^1a^0 = a^{1+0} = a^1 = a.$$

Dividing both sides by a gives

$$a^0 = 1.$$

2.3. **When x is a negative integer.** Consider the case when x is a negative integer, so that $x = -n$ for some positive integer n. For **(E1)** to remain true, we must have

$$a^na^x = a^{n+x} = a^0 = 1.$$

In this case,

$$a^{-n} = \frac{1}{a^n}.$$

2.4. **When x is rational.** Consider the case when $x = \frac{1}{n}$, where n is a positive integer. For **(E2)** to remain true, we must have

$$(a^{1/n})^n = a^{n/n} = a^1 = a.$$

Thus, $a^{1/n}$ is the unique number whose n^{th} power is a; that is,

$$a^{1/n} = \sqrt[n]{a}.$$

Consider the case when $x = \frac{m}{n}$, where m and n are positive integers. Then **(E2)** produces $a^{m/n} = (a^m)^{1/n}$, so

$$a^{m/n} = \sqrt[n]{a^m}.$$

2.5. **When x is irrational.** We now consider the case when x is irrational. This is the hardest step.

Integers are obtained from natural numbers by algebraic considerations (defining subtraction), and rational numbers are obtained from integers by additional algebraic considerations (defining division); however, real numbers are obtained from rationals by geometric considerations (filling in gaps in the number line).

There is an additional property of exponents which is important in this context:

(E3) if $1 < a$ and $r < s$, then $a^r < a^s$

This is true when x is any rational number, and we wish it to remain true for any real number.

We line up all of the rationals by the order relation $<$, and see that there are gaps in the line; so, too, we can line up all of the numbers of the form a^q where q is rational, and see that there are gaps in the line; we hope to fill these gaps by numbers of the form a^x, where x is irrational.

Let $x \in \mathbb{R}$, and let $\lfloor x \rfloor$ denote the *floor* of x; this is the largest integer which is less than or equal to x. We use this to denote rational estimates of a decimal expansion. For example,

- $\lfloor \pi \rfloor = 3$
- $\lfloor 10\pi \rfloor = 31$ $\qquad \dfrac{\lfloor 10\pi \rfloor}{10} = 3.1$
- $\lfloor 10^2\pi \rfloor = 314$ $\qquad \dfrac{\lfloor 10^2\pi \rfloor}{10^2} = 3.14$
- $\lfloor 10^3\pi \rfloor = 3141$ $\qquad \dfrac{\lfloor 10^3\pi \rfloor}{10^3} = 3.141$
- $\lfloor 10^4\pi \rfloor = 31415$ $\qquad \dfrac{\lfloor 10^4\pi \rfloor}{10^4} = 3.1415$
- $\lfloor 10^5\pi \rfloor = 314159$ $\qquad \dfrac{\lfloor 10^5\pi \rfloor}{10^5} = 3.14159$

and so forth. In general, $\dfrac{\lfloor 10^n x \rfloor}{10^n}$ is x to the n^{th} decimal place.

Let x be an irrational number, and define the sequence (x_n) by

$$x_n = \frac{\lfloor 10^{n-1} x \rfloor}{10^{n-1}},$$

so that (x_n) is a sequence of rational numbers; this is a sequence of rational estimates of x of increasing accuracy, and it converges to x:

$$x = \lim_{n\to\infty} x_n.$$

Since x_n is rational, a^{x_n} is defined. Consider the sequence (a^{x_n}); by Property **(E3)**, this is an increasing sequence of real numbers which is bounded above by a^x. Thus, it converges. We define

$$a^x = \lim_{n\to\infty} a^{x_n}.$$

This definition extends the previous definitions in such a way as to preserve properties **(E1)**, **(E2)**, and **(E3)**.

3. Exponential and Logarithmic Functions

3.1. **Exponential Functions.** Let a be a positive real number. We have defined succeed in defining a^x for any real number x. Many of the properties of exponentiation which are relatively obvious for exponents which are positive integers extend to this more general definition. Among these properties are the following.

- **(a)** $a^0 = 1$
- **(b)** $a^1 = a$
- **(c)** $a^{r+s} = a^r a^s$
- **(d)** $(a^r)^s = a^{rs}$
- **(e)** $r < s \Rightarrow a^r < a^s$, if $a > 1$
- **(f)** $r < s \Rightarrow a^r > a^s$, if $0 < a < 1$

If we let x vary through the real numbers, we can view a^x as a function of x with a fixed base a.

Let a be a positive real number, $a \neq 1$. Define a function

$$\exp_a : \mathbb{R} \to \mathbb{R} \text{ given by } \exp_a(x) = a^x.$$

This is called the *base a exponential function.* It satisfies these properties:

- **(a)** $\exp_a(0) = 1$
- **(b)** $\exp_a(1) = a$
- **(c)** $\exp_a(x_1 + x_2) = \exp_a(x_1) \cdot \exp_a(x_2)$
- **(d)** $(\exp_a(x_1))^{x_2} = \exp_a(x_1 x_2)$
- **(e)** $x_1 < x_2 \Rightarrow \exp_a(x_1) < \exp_a(x_2)$, if $a > 1$
- **(f)** $x_1 < x_2 \Rightarrow \exp_a(x_1) > \exp_a(x_2)$, if $0 < a < 1$

Let use assume that $a > 1$; analogous comments apply to the case $0 < a < 1$.

By property **(e)** above, $\exp_a$ is increasing, and therefore, $\exp_a$ is injective. Thus we can construct an inverse for $\exp_a$; the domain of the inverse is the range of the function, so we need to find the range of $\exp_a$.

We wish to show that $a^x \to \infty$ as $x \to \infty$. Let M be a large positive real number; we wish to show that there exists an x so that $a^x \geq M$.

Let $b = a - 1$ so that $b > 0$. Since $a > b$, for any positive integer n we have

$$a^n > b^n = (1 + a)^n = 1 + na + \cdots \geq 1 + na.$$

Let n be so large that $1 + na > M$; then $a^n > M$, which show that

$$\lim_{x \to \infty} a^x = \infty.$$

Since $a^{-x} = \dfrac{1}{a^x}$, we have

$$\lim_{x \to -\infty} a^x = \lim_{x \to \infty} a^{-x} = \lim_{x \to \infty} \frac{1}{a^x} = \frac{1}{\lim\limits_{x \to \infty} a^x} = 0.$$

Thus,

$$\text{range}(\exp_a) = (0, \infty).$$

We now restrict the codomain of $\exp_a$ to its range, making it a injective and surjective, and thus invertible.

3.2. **Logarithmic Functions.** Let a be a positive real number, and set

$$\exp_a : \mathbb{R} \to (0, \infty) \quad \text{be given by } \exp_a(x) = a^x.$$

This function is bijective, and hence invertible.

Let $\log_a$ be the inverse of $\exp_a$; thus

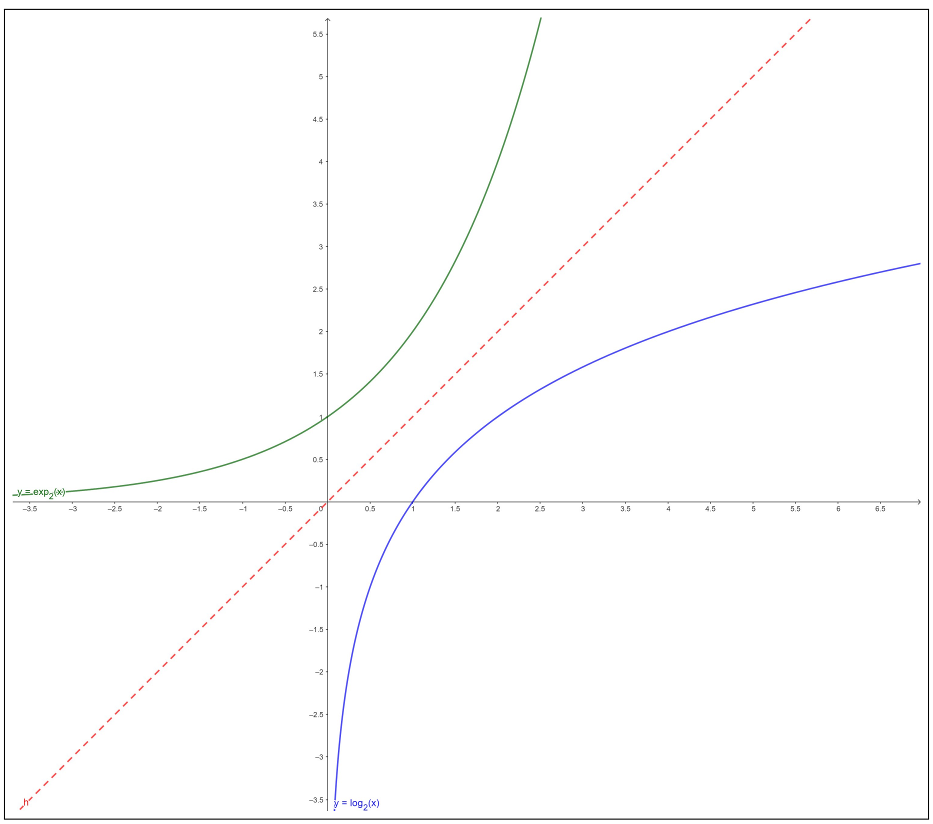
$$\log_a : (0, \infty) \to \mathbb{R} \quad \text{given by} \quad \log_a(x) = y \Leftrightarrow a^y = x.$$

This is called the *base a logarithm.* Since exponential functions convert addition to multiplication, logarithmic function convert multiplication into addition; this was the original motivation for their invention. Moreover, the reader should be aware that the inverse of an increasing function is also increasing.

Logarithms satisfy these properties:

(a) $\log_a(1) = 0$
(b) $\log_a(a) = 1$
(c) $\log_a(x_1 x_2) = \log_a(x_1) + \log_a(x)2$
(d) $\log_a(x^r) = r \log_a(x)$
(e) $x_1 < x_2 \Rightarrow \log_a(x_1) < \log_a(x_2)$, if $a > 1$
(f) $x_1 < x_2 \Rightarrow \log_a(x_1) > \log_a(x_2)$, if $0 < a < 1$

The graphs of $\exp_2$ and $\log_2$ are produced below. As a gets larger, the graph of $\exp_a$ gets steeper for $x > 0$.



4. The Number e

4.1. **Periodic Compound Interest.** Our first examples of exponential functions will be those which compute compound interest. From this, we derive the transcendental number e.

Suppose we invest 1000 dollars at an interest rate of 10 percent compounded annually. The amount we have invested remains the same until one year passes, at which point 10 percent of the amount is added to the total. If we let A_t denote the amount invested after t years, then

- $A_0 = 1000$
- $A_1 = 1000 + (0.1)1000 = 1100$
- $A_2 = 1100 + (0.1)1100 = 1210$
- $A_3 = 1210 + (0.1)1210 = 1331$

We see that the rate at which this grows increases year by year; but the pattern is obscure. It is actually easier to see the pattern if we think more generally.

Let r be the annual interest rate, A_0 the initial investment, and A_t the amount after t years. Then

- $A_1 = A_0 + rA_0 = A_0(1 + r)$
- $A_2 = A_1 + rA_1 = A_1(1 + r) = A_0(1 + r)^2$
- $A_3 = A_2 + rA_2 = A_2(1 + r) = A_0(1 + r)^3$
- $A_t = A_0(1 + r)^t$

Suppose that, instead of compounding annually, we compound quarterly; that is, every three months, or four times per year. Then, the periodic interest rate is the annual rate divided by four.

- $A_{1/4} = A_0 + (\frac{r}{4})A_0 = A_0(1 + \frac{r}{4})$
- $A_{1/2} = A_{1/4} + (\frac{r}{4})A_{1/4} = A_{1/4}(1 + \frac{r}{4}) = A_0(1 + \frac{r}{4})^2$
- $A_1 = A_0(1 + \frac{r}{4})^4$
- $A_t = A_0(1 + \frac{r}{4})^{4t}$

Generalize this further; let k denote the number of periods per year, so that we compound k times per year. Then, there are k times every year when we the amount in the account by $(1 + \frac{r}{k})$; these gives

$$A_t = A_0\left(1 + \frac{r}{k}\right)^{kt},$$

where r is the annual rate, k is the number of periods per year, and A_t is the amount after t years.

The more periods per year, the faster the amount grows, as this table demonstrates. We let the annual rate r be ten percent and the initial investment A_0 be one thousand. We compute the amount after five years for various values of k, to the nearest dollar:

k	A_0	A_1	A_2	A_3	A_4	A_5
1	1000	1100	1210	1331	1464	1611
2	1000	1103	1216	1340	1477	1629
4	1000	1104	1218	1345	1485	1639
12	1000	1105	1220	1348	1489	1645
365	1000	1105	1221	1350	1492	1649
8760	1000	1105	1221	1350	1492	1649

This table demonstrates two facts:

- as k increases, the investment grows faster;
- as k increases, the rate at which the investment grows faster slows down.

4.2. **Continuous Compound Interest.** We wish to define continuously compounded interest as the limit of periodically compounded interest as the k goes to infinity. Thus we fix A_0, r, and t, and attempt to understand the expression

$$\lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt}.$$

To do this, we define a new variable n by $n=\frac{k}{r}$, so that $k=nr$ and $\frac{r}{k}=\frac{1}{n}$. Since r is fixed, n goes to infinity as k goes to infinity. We compute

$$\begin{aligned}\lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt} &= \lim_{n\to\infty} A_0\left(1+\frac{1}{n}\right)^{nrt}\\ &= \lim_{n\to\infty} A_0\left[\left(1+\frac{1}{n}\right)^{n}\right]^{rt}\\ &= A_0\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right]^{rt}.\end{aligned}$$

This computation tells us that continuously compounded interest may be computed using an exponential function whose base is the limit of the sequence $(1+\frac{1}{n})^n$; it can be show that this is an increasing sequence which is bounded above by 3, so it converges. The number it converges to turns out to be so important in mathematics that we give it a special name.

Define

$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}.$$

Then, the equation which computes the amount A_t for continuously compounded interest is

$$A_t = A_0 e^{rt}.$$

We estimate e by computing a few values:

n	$(1+\frac{1}{n})^n$	estimate
1	$(2)^1$	2.000000
2	$(1.5)^2$	2.250000
4	$(1.25)^4$	2.441406
10	$(1.1)^{10}$	2.593742
100	$(1.01)^{100}$	2.704813
1000	$(1.001)^{1000}$	2.716923
10000	$(1.0001)^{10000}$	2.718145
100000	$(1.00001)^{100000}$	2.718268
∞	e	2.718281

5. The Natural Exponential Function

We have previously discussed the meaning of a^x when x is irrational. So, we have a meaning for e^x. It is convenient to rearrange this.

We are given that
$$e = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m.$$
Let x be any real number. Set $n = mx$ so that $\frac{1}{m} = \frac{x}{n}$. Since x is fixed, $n \to \infty$ as $m \to \infty$. Then
$$\begin{aligned} e^x &= \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^x \\ &= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{mx} \\ &= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \end{aligned}$$
The *natural exponential function* is
$$\exp : \mathbb{R} \to (0,\infty) \quad \text{given by} \quad \exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$
That is, $\exp(x) = e^x$. That is, $\exp = \exp_e$.

We wish to compute the derivative of this function. First, we compute
$$\begin{aligned} \frac{d}{dx}e^x &= \frac{d}{dx}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \\ &= \lim_{n\to\infty}\frac{d}{dx}\left(1+\frac{x}{n}\right)^n \qquad \text{leap of faith} \\ &= \lim_{n\to\infty} n\cdot\frac{1}{n}\cdot\left(1+\frac{x}{n}\right)^{n-1} \\ &= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n-1} \\ &= \frac{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n}{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)} \\ &= \frac{e^x}{1} \\ &= e^x. \end{aligned}$$
Thus e^x is a function which is its own derivative. This proof has used the fact that, in this case, the differentiation operator commutes with the limit operator. We will give an alternative derivation shortly.

6. The Natural Logarithm

The *natural logarithm* is the function

$$\log : (0, \infty) \to \mathbb{R} \quad \text{given by} \quad \log(x) = y \Leftrightarrow e^y = x.$$

That is, log is the inverse function of exp, and $\log = \log_e$.

It should be noted that it is not uncommon to let log denote $\log_{10}$, the base ten logarithm. This is called the *common logarithm.* Because of this, the notation $\ln(x)$ is used to mean $\log_e(x)$. We avoid this notation, as (for Calculus and for Statistics), the natural logarithm is far more useful and "natural", and also more "common".

We rewrite the properties of the natural logarithm in this notation:

(a) $\ln(1) = 0$
(b) $\ln(a) = 1$
(c) $\ln(x_1 x_2) = \ln(x_1) + \ln(x)2$
(d) $\ln(x^r) = r \ln(x)$
(e) $x_1 < x_2 \Rightarrow \ln(x_1) < \ln(x_2)$, if $a > 1$
(f) $x_1 < x_2 \Rightarrow \ln(x_1) > \ln(x_2)$, if $0 < a < 1$

It is now convenient to derive the change of base formula for logarithms. We know that $\log_a(x) = y$ if and only if $a^y = x$. So, $\ln(a^y) = \ln(x)$, so $y \ln(a) = \ln(x)$. Therefore,

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}.$$

One may use this formula with a scientific calculator to compute logarithms in any base.

We now compute the derivative of the natural logarithm, using the fact that $\frac{d}{dx} e^x = e^x$. We let $y = \ln(x)$, and use implicit differentiation to find $\frac{dy}{dx}$. We have

$$\begin{aligned} y = \ln(x) \quad &\Leftrightarrow \quad e^y = x \\ &\Leftrightarrow \quad \frac{d}{dx} e^y = \frac{d}{dx} x \\ &\Leftrightarrow \quad e^y \frac{dy}{dx} = 1 \\ &\Leftrightarrow \quad \frac{dy}{dx} = \frac{1}{e^y} \\ &\Leftrightarrow \quad \frac{dy}{dx} = \frac{1}{x} \end{aligned}$$

That is, the derivative of $\ln(x)$ is $\frac{1}{x}$.

7. Alternative Derivation of the Derivatives

As promised, we give an alternative derivation of the derivative of exp and log; in this case, start with log.

This derivation has these ingredients:

- $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$
- $e^x = \lim_{n\to\infty}(1+\dfrac{x}{n})^n$
- $\ln(x) = y \Leftrightarrow e^y = x$
- $\ln(x_1) - \ln(x_2) = \ln\left(\dfrac{x_1}{x_2}\right)$

Let $f(x) = \ln(x)$. Then, by definition of derivative,

$$\begin{aligned}
\frac{df}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} && \text{by definition of derivative}\\
&= \lim_{h\to 0}\frac{\ln(x+h)-\ln(x)}{h} && \text{since } f(x)=\ln(x)\\
&= \lim_{h\to 0}\frac{1}{h}\ln\left(\frac{x+h}{x}\right) && \text{by a property of logarithm}\\
&= \lim_{h\to 0}\frac{1}{h}\ln\left(1+\frac{h}{x}\right) &&\\
&= \lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}} && \text{by a property of logarithm}\\
&= \lim_{h\to 0}\ln\left(1+\frac{1/x}{1/h}\right)^{\frac{1}{h}} &&\\
&= \ln\lim_{n\to\infty}\left(1+\frac{1/x}{n}\right)^{n} && \text{since limit commutes with logarithm}\\
&= \ln(e^{1/x}) && \text{by definition of } e^u\\
&= \frac{1}{x}
\end{aligned}$$

Now, to obtain the derivative of e^x, we have

$$\begin{aligned}
y = e^x \quad &\Leftrightarrow \quad \ln(y) = x\\
&\Leftrightarrow \quad \frac{d}{dx}\ln(y) = \frac{d}{dx}x\\
&\Leftrightarrow \quad \frac{1}{y}\frac{dy}{dx} = 1\\
&\Leftrightarrow \quad \frac{dy}{dx} = y\\
&\Leftrightarrow \quad \frac{dy}{dx} = e^x
\end{aligned}$$

That is, the derivative of e^x is e^x.

8. Derivatives of Exp and Log in Other Bases

Let a be a positive real number. Using properties of exponentials and logarithms, we see that

$$\exp_a(x) = a^x = \exp(\log(a^x)) = \exp(x\log(a)).$$

We use this to compute the derivative of a^x.

$$\begin{aligned}
\frac{d}{dx}a^x &= \frac{d}{dx}\exp(x\log(a)) \\
&= \exp(\log(a)) \cdot \log(a) && \text{by the Chain Rule} \\
&= \ln(a)a^x
\end{aligned}$$

Thus, the derivative of a^x is $\ln(a)a^x$.

Finally, we produce the derivative of logs to other bases.

$$\begin{aligned}
y = \log_a(x) \quad &\Leftrightarrow \quad a^y = x \\
&\Leftrightarrow \quad \frac{d}{dx}a^y = \frac{d}{dx}x \\
&\Leftrightarrow \quad \ln(a)a^y\frac{dy}{dx} = 1 \\
&\Leftrightarrow \quad \frac{dy}{dx} = \frac{1}{\ln(a)a^y} \\
&\Leftrightarrow \quad \frac{dy}{dx} = \frac{1}{\ln(a)x}
\end{aligned}$$

Thus, the derivative of $\log_a(x)$ is $\dfrac{1}{\ln(a)x}$.

Department of Mathematics, Sonoran Science Academy
Email address: `pbailey@sonoranschools.org`